MATHEMATICS II -may 2004

1. Examine the convergence of the following series
$$\sum_{n=1}^{\infty} \frac{1}{a^{b\ln n + c\ln^2 n}},$$
where is $a > 0$, $a \neq 1$, $b, c \in \mathbb{R}$,

2. Find volume of the following region
$$V = \{(x,y,z):\ (\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2})^2 \leq y\} \quad (a,b,c > 0).$$

3. If $\alpha, p \in \mathbb{R}$, $0 < \alpha < \pi$, $0 < |p| < 1$, calculate the given convergent integral
$$\int_0^{\infty} \frac{x^{-p}dx}{x^2 + 2x\cos\alpha + 1}.$$

MATEMATIKA II -maj 2004

1. Ako je $a > 0$, $a \neq 1$, $b, c \in \mathbb{R}$, ispitati konvergenciju reda $\sum\limits_{n=1}^{\infty} \frac{1}{a^{b\ln n + c\ln^2 n}}$.

2. Naći zapreminu oblasti
$$V = \{(x,y,z):\ (\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2})^2 \leq y\} \quad (a,b,c > 0).$$

3. Ako $\alpha, p \in \mathbb{R}$, tako da $0 < \alpha < \pi$, $0 < |p| < 1$, izračunati konvergentan integral
$$\int_0^{\infty} \frac{x^{-p}dx}{x^2 + 2x\cos\alpha + 1}.$$

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Rešenje:

1. Iz osobina logaritma imamo da je
$$\alpha_n = a^{-b\ln n - c\ln^2 n} = n^{-b\ln a - c\ln a \cdot \ln n}.$$

Ako je $c\ln a > 0$, tada postoji $n_0 \in \mathbb{N}$, tako da za sve $n \geq n_0$ važi $b\ln a + c\ln a \cdot \ln n \geq d = b\ln a + c\ln a \cdot \ln n_0 > 1$ (jer $\lim\limits_{n\to\infty} \ln n = \infty$, tj. $\lim\limits_{n\to\infty} b\ln a + c\ln a \cdot \ln n = \infty$), pa je
$$\alpha_n \leq \frac{1}{n^d},$$
te po uporednom kriterijumu iz konvergencije reda $\sum\limits_{n=1}^{\infty} \frac{1}{n^d}$ sledi i konvergencija reda $\sum\limits_{n=1}^{\infty} \alpha_n$.

Ako je $c\ln a <$, tada postoji $n_0 \in \mathbb{N}$, tako da za sve $n \geq n_0$ važi $b\ln a + c\ln a \cdot \ln n \leq d = b\ln a + c\ln a \cdot \ln n_0 < 0$ (jer $\lim\limits_{n\to\infty} b\ln a + c\ln a \cdot \ln n = -\infty$), pa je
$$\alpha_n \geq n^{-d},$$

te po uporednom kriterijumu iz divergencije reda $\sum\limits_{n=1}^{\infty} n^{-d}$ sledi i divergencija reda $\sum\limits_{n=1}^{\infty} \alpha_n$.

Ako je $c = 0$, tada je

$$\alpha_n = \frac{1}{n^{b \ln a}},$$

pa red $\sum\limits_{n=1}^{\infty} \alpha_n$ konvergira samo ako je $b \ln a > 1$, dok u suprotnom divergira.

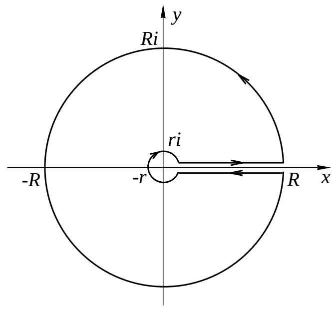
2. $x = a\rho \cos\varphi \sin\theta$, $y = b\rho \sin\varphi \sin\theta$, $z = c\rho\cos\theta$,

$$J = \begin{vmatrix} a\cos\varphi\sin\theta & b\sin\varphi\sin\theta & c\cos\theta \\ -a\rho\sin\varphi\sin\theta & b\rho\cos\varphi\sin\theta & 0 \\ a\rho\cos\varphi\cos\theta & b\rho\sin\varphi\cos\theta & -c\rho\sin\theta \end{vmatrix} = -abc\rho^2\sin\theta.$$

$(\rho^2)^2 = b\rho\sin\varphi\sin\theta \Rightarrow \rho = \sqrt[3]{b\sin\varphi\sin\theta} \Rightarrow \sin\varphi \geq 0 \Rightarrow \varphi \in [0,\pi]$, $\theta \in [0,\pi]$, $\rho \in [0, \sqrt[3]{b\sin\varphi\sin\theta}]$.

$V = \iiint\limits_V dxdydz = \iiint\limits_{V'} |J| d\rho d\varphi d\theta = abc \int\limits_0^{\pi}\int\limits_0^{\pi} \int\limits_0^{\sqrt[3]{b\sin\varphi\sin\theta}} \rho^2 \sin\theta d\rho d\theta d\varphi = \frac{\pi}{3} ab^2 c.$

3. Posmatrajmo kompleksni integral $\int\limits_L \frac{z^{-p}}{z^2+2z\cos\alpha+1} dz$, gde je L kontura data na slici.



Uzmimo onu granu funkcije $z^{-p} = e^{-p(\ln|z| + i\arg z + 2k\pi i)}$ za koju unutar date konture L važi $\operatorname{Arg} z = \arg z + 2k\pi \in (0, 2\pi)$, tj. $\arg z \in (0, 2\pi)$. U tom slučaju će biti i

$z = x \in \mathbb{R}^+ \Rightarrow z^{-p} = e^{-p(\ln x + i\cdot 0)} = e^{-p\ln x} = x^{-p}$,
$z = xe^{i2\pi}, x \in \mathbb{R}^+ \Rightarrow z^{-p} = e^{-p(\ln x + i2\pi)} = e^{-p\ln x} e^{-2p\pi i} = x^{-p}e^{-2p\pi i}$,
$z = -e^{-i\alpha} \Rightarrow z^{-p} = e^{-p(\ln 1 + i(\pi-\alpha))} = e^{-pi(\pi-\alpha)}$,
$z = -e^{i\alpha} \Rightarrow z^{-p} = e^{-p(\ln 1 + i(\pi+\alpha))} = e^{-pi(\pi+\alpha)}$.

$z^2 + 2z\cos\alpha + 1 = 0 \Leftrightarrow z_{1/2} = -e^{\pm i\alpha}$, $\alpha \in (0,\pi) \Rightarrow z_{1/2} \in \operatorname{int} L$. Odatle, sledi da su $z_{1/2}$ polovi prvog reda funkcije $f(z) = \frac{z^{-p}}{z^2+2z\cos\alpha+1}$, pa je $I = \int\limits_L f(z)dz = 2\pi i\,(\operatorname*{Res}\limits_{z=z_1} f(z) + \operatorname*{Res}\limits_{z=z_2} f(z)) = 2\pi i\cdot(\frac{z^{-p}}{2z+2\cos\alpha}\Big|_{z=z_1} + \frac{z^{-p}}{2z+2\cos\alpha}\Big|_{z=z_2})$
$= 2\pi i\cdot\frac{1}{2}(\frac{(-e^{i\alpha})^{-p}}{-i\sin\alpha} + \frac{(-e^{-i\alpha})^{-p}}{i\sin\alpha}) = \frac{\pi}{\sin\alpha}[(-e^{-i\alpha})^{-p} - (-e^{i\alpha})^{-p}] = \frac{\pi}{\sin\alpha}[e^{-pi(\pi-\alpha)} - e^{-pi(\pi+\alpha)}] = \frac{\pi}{\sin\alpha} e^{-p\pi i}(e^{ip\alpha} - e^{-ip\alpha})$
$= \frac{\pi}{\sin\alpha} e^{-p\pi i}\cdot 2i\sin p\alpha = 2\pi i e^{-p\pi i}\frac{\sin p\alpha}{\sin\alpha}$.

Sa druge strane je $I = \int\limits_r^R \frac{x^{-p}}{x^2+2x\cos\alpha+1}dx + \int\limits_0^{2\pi} \frac{R^{-p}e^{-pit}}{R^2e^{2it}+2Re^{it}\cos\alpha+1} Rie^{it}dt + \int\limits_R^r \frac{x^{-p}e^{-pi2\pi}}{x^2e^{4i\pi}+2xe^{2i\pi}\cos\alpha+1} e^{2i\pi}dx + \int\limits_{2\pi}^0 \frac{r^{-p}e^{-pit}}{r^2e^{2it}+2re^{it}\cos\alpha-}$
$I_1 + I_2 + I_3 + I_4$.

$|I_2| \leq \int\limits_0^{2\pi} \frac{|R^{-p}e^{-pit}|}{||R^2e^{2it}| - |2Re^{it}\cos\alpha| - 1|} |Rie^{it}| dt = \int\limits_0^{2\pi} \frac{R^{-p}}{R^2 - 2R|\cos\alpha| - 1} R dt$
$= 2\pi\cdot \frac{R^{1-p}}{R^2-2R|\cos\alpha|-1} \Rightarrow \lim\limits_{R\to\infty} I_2 = 0$, jer $2\pi\frac{R^{1-p}}{R^2-2R|\cos\alpha|-1} \sim 2\pi R^{-1-p}$, $R\to\infty$, $-1-p<0$.

Slično, $|I_4| \leq \int\limits_0^{2\pi} \frac{|r^{-p}e^{-pit}|}{||r^2e^{2it}| - |2re^{it}\cos\alpha| - 1|} |rie^{it}| dt = \int\limits_0^{2\pi} \frac{r^{-p}}{1-2r|\cos\alpha|-r^2} r dt = 2\pi\ \frac{r^{1-p}}{1-2r|\cos\alpha|-r^2} \Rightarrow \lim\limits_{r\to 0} I_4 = 0$, jer $2\pi\frac{r^{1-p}}{1-2r|\cos\alpha|-r^2} \to 0$, $r\to 0$, $1-p>0$.

Dakle, kada $r\to 0$, $R\to\infty$, tada je
$(1 - e^{-2pi\pi}) \int\limits_0^{\infty} \frac{x^{-p}dx}{x^2+2x\cos\alpha+1} = 2\pi i\cdot\ e^{-p\pi i}\frac{\sin p\alpha}{\sin\alpha}$, tj.

$$\int\limits_0^{\infty} \frac{x^{-p}dx}{x^2+2x\cos\alpha+1} = \frac{2\pi i}{1-e^{-2p\pi i}}\cdot e^{-p\pi i}\frac{\sin p\alpha}{\sin\alpha} = \frac{\pi}{\sin p\pi}\cdot\frac{\sin p\alpha}{\sin\alpha}.$$