# MATHEMATICS I -may 2004

1. a) Prove that

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\leq j<i\leq n} (a_i - a_j).$$

b) Prove that exists vector $x \in \mathbb{R}^n$, such that vectors $x$, $Dx$, ... $D^{n-1}x$ compose base in $\mathbb{R}^n$, where

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \cdots & & \vdots \\ 0 & \cdots & & d_n \end{bmatrix} \in \mathbb{R}^{n,n}, \quad d_i \neq d_j, i \neq j, i, j \in \{1, ..., n\}.$$

2.

$$a) \quad \lim_{n\to\infty} \sin(\sin(...(\sin\alpha)...)) = ? \ \ (\alpha \in \mathbb{R}). \qquad b) \quad \lim_{n\to\infty} \sin(\pi\sqrt{n^2+1}) = ?$$

3. Find primitive functions of:

$$a) \quad f(x) = \frac{1+\sin x}{1+\cos x} e^x, \quad 0 < x < \pi;$$

$$b) \quad f(x) = \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x}, \quad x \in \mathbb{R}.$$

# MATEMATIKA I -maj 2004

1. a) Dokazati da je

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\leq j<i\leq n} (a_i - a_j).$$

b) Dokazati da postoji vektor $x \in \mathbb{R}^n$, takav da vektori $x$, $Dx$, ... $D^{n-1}x$ čine bazu za $\mathbb{R}^n$, gde je

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \cdots & & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix} \in \mathbb{R}^{n,n}, \quad d_i \neq d_j, i \neq j, i, j \in \{1, ..., n\}.$$

2.

$$a) \quad \lim_{n\to\infty} \sin(\sin(...(\sin\alpha)...)) = ? \ \ (\alpha \in \mathbb{R}). \qquad b) \quad \lim_{n\to\infty} \sin(\pi\sqrt{n^2+1}) = ?$$

3. Naći primitivne funkcije od:

$$a) \quad f(x) = \frac{1+\sin x}{1+\cos x} e^x, \quad 0 < x < \pi;$$

$$b) \quad f(x) = \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x}, \quad x \in \mathbb{R}.$$

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**Rešenje**:

**1.** Dokažimo matematičkom indukcijom da je prethodna Vandermondova determinanta jednaka $V_n = V_n(a_1, a_2, ..., a_n) = \prod\limits_{1\leq j<i\leq n}(a_i - a_j)$. Za $n = 2$ je $V_2 = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1 = \prod\limits_{1\leq j<i\leq 2}(a_i - a_j)$, što je tacno. Pretpostavimo da je tvrđenje tačno za determinantu reda $n$ (i bilo koje $a_1, ..., a_n$) i pokažimo da je tačno za determinantu reda $n + 1$.

$$V_{n+1} = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} & a_1^n \\ 1 & a_2 & \cdots & a_2^{n-1} & a_2^n \\ \vdots & & \cdots & & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} & a_n^n \end{vmatrix} = \begin{vmatrix} 1 & a_1 - a_n & \cdots & a_1^{n-1} - a_1^{n-2}a_n & a_1^n - a_1^{n-1}a_n \\ 1 & a_2 - a_n & \cdots & a_2^{n-1} - a_2^{n-2}a_n & a_2^n - a_2^{n-1}a_n \\ \vdots & & \cdots & & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{n+1}(a_1 - a_n)(a_2 - a_n)...(a_{n-1} - a_n)\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & & \cdots & & \vdots \\ 1 & a_{n-1} & \cdots & a_{n-1}^{n-2} & a_{n-1}^{n-1} \end{vmatrix} = (a_n - a_1)(a_n - a_2)...(a_n - a_{n-1})V_n$$

$$= \prod_{1\leq j<i\leq n+1}(a_i - a_j).$$

Lako se vidi (dokaz indukcijom) da je za $k \in \mathbb{N}$:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \cdots & & \vdots \\ 0 & \cdots & & d_n^k \end{bmatrix} \quad \Rightarrow \quad \sum_{k=0}^{n-1}\alpha_k D^k x = \begin{bmatrix} \sum\limits_{k=0}^{n-1}\alpha_k d_1^k x_1 \\ \sum\limits_{k=0}^{n-1}\alpha_k d_2^k x_2 \\ \vdots \\ \sum\limits_{k=0}^{n-1}\alpha_k d_n^k x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{matrix} \alpha_0 x_1 + \alpha_1 d_1 x_1 + ... + \alpha_{n-1} d^{n-1} x_1 = 0 \\ \alpha_0 x_2 + \alpha_1 d_1 x_2 + ... + \alpha_{n-1} d^{n-1} x_2 = 0 \\ \vdots \\ \alpha_0 x_n + \alpha_1 d_1 x_n + ... + \alpha_{n-1} d^{n-1} x_n = 0 \end{matrix} \Leftrightarrow \begin{bmatrix} x_1 & x_1 d_1 & \cdots & x_1 d_1^{n-1} \\ x_2 & x_2 d_2 & \cdots & x_2 d_2^{n-1} \\ \vdots & & \cdots & \vdots \\ x_n & x_n d_n & \cdots & x_n d_n^{n-1} \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Determinanta sistema je $D = x_1x_2...x_n \prod\limits_{1\leq j<i\leq n}(d_i - d_j)$, pa kako je $d_i \neq d_j$, $i \neq j$, $i, j \in \{1, ..., n\}$ uzimajući $x_i \neq 0$, $i \in \{1, ..., n\}$ ona je različita od nule te sistem ima trivijalno rešenje $\alpha_i = 0$, $i \in \{0, 1, ..., n-1\}$, odnosno $x$, $Dx$, ... $D^{n-1}x$ su linearno nezavisni vektori, a kako je $dim(\mathbb{R}^n) = n$, oni čine bazu.

**2.** a) Treba naći $\lim\limits_{n\to\infty} a_n$ gde je $a_1 = \sin\alpha$ i $a_{n+1} = \sin a_n$, $n \in \mathbb{N}$.

Pretpostavimo da je $\sin\alpha > 0$.

Imamo $a_2 = \sin a_1 \in [0, 1] \subset [0, \frac{\pi}{2})$. Dokažimo indukcijom da je $a_n \in [0, 1]$. Za $n = 2$ smo pokazali. Pretpostavimo da je tačno tvrđenje tačno za $n$, tj. $0 \leq a_n \leq 1$, odakle iz monotonije sin sledi $0 = \sin 0 \leq \sin a_n \leq \sin 1 \leq 1$, odnosno $a_{n+1} \in [0, 1]$, te je tvrđenje tačno za $n + 1$.

Pokažimo indukcijom da je niz monotono nerastući . Za $n = 1$ je $a_2 = \sin a_1 \leq a_1$. Pretpostavimo da je $a_{n-1} \leq a_n$. Odatle znajući da je $a_n \in [0, 1] \subset [0, \frac{\pi}{2}]$, $n \in \mathbb{N}$ iz monotonosti sin sledi $a_n = \sin a_{n-1} \leq \sin a_n = a_{n+1}$.

Dakle, niz je monotono nerastući i ograničen sa donje strane te konvergira ka svome infimumu $i$. Iz $0 \leq a_{n+1} = \sin a_n$, sledi $0 \leq i = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sin a_n = \sin \lim\limits_{n\to\infty} a_n = \sin i \leq 1$, pa je $i = 0$. Analogno se i za slučaj $\sin\alpha \leq 0$ dokazuje da je $i = 0$.

b) Za $a_n = \sin\pi\sqrt{n^2+1}$ je $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} |\sin\pi(\sqrt{n^2+1} - n + n)| = \lim\limits_{n\to\infty} |(-1)^n \sin\pi(\sqrt{n^2+1} - n)| = \lim\limits_{n\to\infty} \sin\frac{\pi}{\sqrt{n^2+1}+n} = \sin\lim\limits_{n\to\infty}\frac{\pi}{\sqrt{n^2+1}+n} = 0$. Kako je $\lim\limits_{n\to\infty} |a_n| = 0$, to sledi da je i $\lim\limits_{n\to\infty} a_n = 0$.

**3.** a) Parcijalnom integracijom ($u = \frac{1+\sin x}{1+\cos x}$, $dv = e^x dx$, $du = \frac{1+\cos x+\sin x}{(1+\cos x)^2}$, $v = e^x$) imamo

$F(x) = \int \frac{1+\sin x}{1+\cos x}e^x dx = \frac{1+\sin x}{1+\cos x}e^x - \int e^x \frac{1+\cos x+\sin x}{(1+\cos x)^2}dx = \frac{1+\sin x}{1+\cos x}e^x - \int \frac{e^x}{1+\cos x}dx - \int \frac{e^x \sin x}{(1+\cos x)^2}dx.$

Ponovnom parcijalnom integracijom ($u = e^x$, $dv = \frac{\sin x}{(1+\cos x)^2}$, $du = e^x dx$, $v = \frac{1}{1+\cos x}$) imamo

$F(x)=\frac{1+\sin x}{1+\cos x}e^x-\int\frac{e^x}{1+\cos x}dx-\frac{e^x}{1+\cos x}+\int\frac{e^x}{1+\cos x}dx+c=\frac{\sin x}{1+\cos x}e^x+c.$

b) $F(x)=\int\frac{\sin^2 x\cos^2 x}{\sin^8 x+\cos^8 x}dx=\int\frac{\frac{1}{4}\sin^2 2x}{(\sin^4 x+\cos^4 x)^2-2\sin^4 x\cos^4 x}dx=\int\frac{\frac{1}{4}\sin^2 2x}{((\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x)^2-\frac{1}{8}\sin^4 2x}dx$
$=\int\frac{\frac{1}{4}\sin^2 2x}{(1-\frac{1}{2}\sin^2 2x)^2-\frac{1}{8}\sin^4 2x}dx=\int\frac{\sin^2 2x}{8-8\sin^2 2x+\sin^4 2x}dx=\int\frac{\sin^2 2x}{8\cos^2 2x+\sin^4 2x}dx=\int\frac{\frac{\sin^2 2x}{\cos^2 2x}}{\frac{8}{\cos^2 2x}+\frac{\sin^4 2x}{\cos^4 2x}}\frac{2}{\cos^2 2x}dx$
$=\int\frac{\operatorname{tg}^2 2x}{8(\operatorname{tg}^2 2x+1)+\operatorname{tg}^4 2x}d(\operatorname{tg}2x)=\int\frac{t^2}{t^4+8t^2+8}dt=\frac{\sqrt{2}+1}{2}\int\frac{dt}{t^2+4+2\sqrt{2}}-\frac{\sqrt{2}-1}{2}\int\frac{dt}{t^2+4-2\sqrt{2}}$
$=\frac{\sqrt{2+\sqrt{2}}}{4}\operatorname{arctg}\frac{t}{\sqrt{4+\sqrt{2}}}-\frac{\sqrt{2-\sqrt{2}}}{4}\operatorname{arctg}\frac{t}{\sqrt{4-\sqrt{2}}}+c$
$=\frac{\sqrt{2+\sqrt{2}}}{4}\operatorname{arctg}\frac{\operatorname{tg}2x}{\sqrt{4+\sqrt{2}}}-\frac{\sqrt{2-\sqrt{2}}}{4}\operatorname{arctg}\frac{\operatorname{tg}2x}{\sqrt{4-\sqrt{2}}}+c_k,\ x\in(-\frac{\pi}{4}+\frac{k\pi}{2},\frac{\pi}{4}+\frac{k\pi}{2}),\ k\in\mathbb{Z}.$

Ukoliko se želi primitivna funkcija nad celim $\mathbb{R}$, onda se ona u tačkama prekida $x=\frac{\pi}{4}+\frac{k\pi}{2}$, $k\in\mathbb{Z}$ mora dodefinisati da bude neprekidna. Tada je

$F(\frac{\pi}{4}+\frac{k\pi}{2})=\lim\limits_{x\to(\frac{\pi}{4}+\frac{k\pi}{2})^-}F(x)=\frac{\sqrt{2+\sqrt{2}}}{4}\frac{\pi}{2}-\frac{\sqrt{2-\sqrt{2}}}{4}\frac{\pi}{2}+c_k=\lim\limits_{(\frac{\pi}{4}+\frac{k\pi}{2})^+}F(x)=-\frac{\sqrt{2+\sqrt{2}}}{4}\frac{\pi}{2}+\frac{\sqrt{2-\sqrt{2}}}{4}\frac{\pi}{2}+c_{k+1}$, a otuda je $c_k=(\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}})\frac{\pi}{4}+c_0$.