MATHEMATICS I -May 2007 Čanj

1. Prove

$$(\overrightarrow{a}\times\overrightarrow{b})\cdot(\overrightarrow{c}\times\overrightarrow{d})=\begin{vmatrix}\overrightarrow{a}\cdot\overrightarrow{c} & \overrightarrow{a}\cdot\overrightarrow{d}\\ \overrightarrow{b}\cdot\overrightarrow{c} & \overrightarrow{b}\cdot\overrightarrow{d}\end{vmatrix},$$

for $\overrightarrow{a},\overrightarrow{b},\overrightarrow{c},\overrightarrow{d}\in\mathbb{R}^3$.

2. Examine detaily the function

$$f(x)=\operatorname{arctg}\frac{1}{\sqrt{|x^2-1|}},$$

and draw its graph.

3. Find the integral:

$$I=\int\frac{dx}{1+\sqrt{x}+\sqrt{x+1}}.$$

MATEMATIKA I -maj 2007 Čanj

1. Dokazati

$$(\overrightarrow{a}\times\overrightarrow{b})\cdot(\overrightarrow{c}\times\overrightarrow{d})=\begin{vmatrix}\overrightarrow{a}\cdot\overrightarrow{c} & \overrightarrow{a}\cdot\overrightarrow{d}\\ \overrightarrow{b}\cdot\overrightarrow{c} & \overrightarrow{b}\cdot\overrightarrow{d}\end{vmatrix},$$

za $\overrightarrow{a},\overrightarrow{b},\overrightarrow{c},\overrightarrow{d}\in\mathbb{R}^3$.

2. Detaljno ispitati funkciju

$$f(x)=\operatorname{arctg}\frac{1}{\sqrt{|x^2-1|}},$$

i nacrtati njen grafik.

3. Naći integral:

$$I=\int\frac{dx}{1+\sqrt{x}+\sqrt{x+1}}.$$

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Rešenje:

1. Dokažimo identitet $\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c})=(\overrightarrow{a}\cdot\overrightarrow{c})\overrightarrow{b}-(\overrightarrow{a}\cdot\overrightarrow{b})\overrightarrow{c}$.

$$\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c})=(a_1,a_2,a_3)\times\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}=\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\ a_1&a_2&a_3\\ \begin{vmatrix}b_2&b_3\\c_2&c_3\end{vmatrix}&\begin{vmatrix}b_3&b_1\\c_3&c_1\end{vmatrix}&\begin{vmatrix}b_1&b_2\\c_1&c_2\end{vmatrix}\end{vmatrix}$$

$$=(a_2\begin{vmatrix}b_1&b_2\\c_1&c_2\end{vmatrix}-a_3\begin{vmatrix}b_3&b_1\\c_3&c_1\end{vmatrix},a_3\begin{vmatrix}b_2&b_3\\c_2&c_3\end{vmatrix}-a_1\begin{vmatrix}b_1&b_2\\c_1&c_2\end{vmatrix},a_1\begin{vmatrix}b_3&b_1\\c_3&c_1\end{vmatrix}-a_2\begin{vmatrix}b_2&b_3\\c_2&c_3\end{vmatrix})$$

$$=(a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3,a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2)$$

$$=(a_2c_2+a_3c_3,a_3c_3+a_1c_1,a_1c_1+a_2c_2)(b_1,b_2,b_3)-(a_2b_2+a_3b_3,a_3b_3+a_1b_1,a_1b_1+a_2b_2)(c_1,c_2,c_3)$$

$$=(a_1c_1+a_2c_2+a_3c_3)(b_1,b_2,b_3)-(a_1b_1+a_2b_2+a_3b_3)(c_1,c_2,c_3)=(\overrightarrow{a}\cdot\overrightarrow{c})\overrightarrow{b}-(\overrightarrow{a}\cdot\overrightarrow{b})\overrightarrow{c}.$$

Koristeći osobine $(\overrightarrow{m}\times\overrightarrow{n})\overrightarrow{p}=(\overrightarrow{n}\times\overrightarrow{p})\overrightarrow{m}$, $(\overrightarrow{m}+\overrightarrow{n})\overrightarrow{p}=\overrightarrow{m}\,\overrightarrow{p}+\overrightarrow{n}\,\overrightarrow{p}$, $(\alpha\overrightarrow{m})\overrightarrow{n}=\alpha(\overrightarrow{m}\,\overrightarrow{n})$, $\overrightarrow{m}\,\overrightarrow{n}=\overrightarrow{n}\,\overrightarrow{m}$, kao i dati identitet, imamo:

$$(\overrightarrow{a}\times\overrightarrow{b})\cdot(\overrightarrow{c}\times\overrightarrow{d})=[\overrightarrow{b}\times(\overrightarrow{c}\times\overrightarrow{d})]\cdot\overrightarrow{a}=[(\overrightarrow{b}\cdot\overrightarrow{d})\overrightarrow{c}-(\overrightarrow{b}\cdot\overrightarrow{c})\overrightarrow{d}]\cdot\overrightarrow{a}$$

$$=(\overrightarrow{b}\cdot\overrightarrow{d})(\overrightarrow{c}\cdot\overrightarrow{a})-(\overrightarrow{b}\cdot\overrightarrow{c})(\overrightarrow{d}\cdot\overrightarrow{a})=(\overrightarrow{a}\cdot\overrightarrow{c})(\overrightarrow{b}\cdot\overrightarrow{d})-(\overrightarrow{a}\cdot\overrightarrow{d})(\overrightarrow{b}\cdot\overrightarrow{c})=\begin{vmatrix}\overrightarrow{a}\cdot\overrightarrow{c} & \overrightarrow{a}\cdot\overrightarrow{d}\\ \overrightarrow{b}\cdot\overrightarrow{c} & \overrightarrow{b}\cdot\overrightarrow{d}\end{vmatrix}.$$

2. $\mathcal{D}_f = \mathbb{R}\setminus\{-1,1\} \;\Rightarrow\; f(x) = \operatorname{arctg}\frac{1}{\sqrt{|x^2-1|}} = \begin{cases} \operatorname{arctg}\frac{1}{\sqrt{x^2-1}}, & x\in(-\infty,-1)\cup(1,\infty) \\ \operatorname{arctg}\frac{1}{\sqrt{1-x^2}}, & x\in(-1,1) \end{cases}.$

Iz $f(x)=f(-x)$, $x\in\mathcal{D}_f$ sledi da je f parna funkcija, pa ćemo je razmatrati samo na intervalu $[0,\infty)$.

$x>0 \Rightarrow f(x)>0$ ($t>0 \Leftrightarrow \operatorname{arctg} t>0$), $f(0)=\frac{\pi}{4}$.

$\lim\limits_{x\to 1+} f(x) = \frac{\pi}{2} = \lim\limits_{x\to 1-} f(x)$. Grafik funkcije ima horizontalnu asimptotu kada $x\to\infty$, i to je $x-$osa jer je $\lim\limits_{x\to\infty} f(x) = \operatorname{arctg}\lim\limits_{x\to\infty}\frac{1}{\sqrt{x^2-1}} = 0$.

$$f'(x) = \begin{cases} \frac{1}{1+\frac{1}{x^2-1}}\cdot(-\frac{1}{2})(x^2-1)^{-\frac{3}{2}}\cdot 2x, & x\in(1,\infty) \\ \frac{1}{1+\frac{1}{1-x^2}}\cdot(-\frac{1}{2})(1-x^2)^{-\frac{3}{2}}\cdot(-2x), & x\in[0,1) \end{cases} = \begin{cases} -\frac{1}{x\sqrt{x^2-1}}, & x\in(1,\infty) \\ \frac{x}{(2-x^2)\sqrt{1-x^2}}, & x\in[0,1) \end{cases}.$$

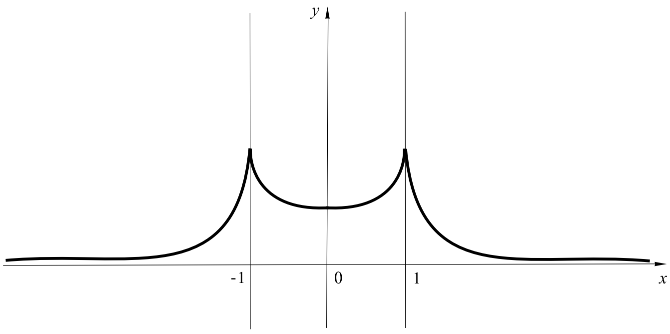
$\lim\limits_{x\to 1+} f'(x) = -\infty$, $\lim\limits_{x\to 1-} f'(x) = -\infty$.

$x\in(1,\infty) \Rightarrow f'(x)<0 \Rightarrow f \nearrow$, $x\in(0,1) \Rightarrow f'(x)>0 \Rightarrow f \searrow$, $f'(0)=0$.

$$f''(x) = \begin{cases} \frac{2x^2-1}{x^2\sqrt{(x^2-1)^3}}, & x\in(1,\infty) \\ \frac{2+x^2-2x^4}{(2-x^2)^2\sqrt{(1-x^2)^3}}, & x\in[0,1) \end{cases},$$

$x\in(1,\infty) \Rightarrow f''(x)>0 \;\Rightarrow\; f \smile$,

$x\in(0,1) \Rightarrow f''(x)>0 \;\Rightarrow\; f \smile$.



3. Integral je definisan za $x\geq 0$, $x+1\geq 0$, tj. za $x\geq 0$. Tada racionalizacijom imamo

$I = \int\frac{1+\sqrt{x}-\sqrt{x+1}}{(1+\sqrt{x})^2-(\sqrt{x+1})^2}dx = \int\frac{1+\sqrt{x}-\sqrt{x+1}}{2\sqrt{x}}dx = \frac{1}{2}\int(\frac{1}{\sqrt{x}}+1-\sqrt{\frac{x+1}{x}})dx = \frac{1}{2}(2\sqrt{x}+x-\int\sqrt{\frac{x+1}{x}}dx)$.

Integral $J=\int\sqrt{\frac{x+1}{x}}dx$ možemo rešiti smenom $x=\operatorname{sh}^2 u$, $0<u=\operatorname{Arsh}\sqrt{x}=\ln(\sqrt{x}+\sqrt{(\sqrt{x})^2+1}) = \ln(\sqrt{x+1}+\sqrt{x})$, $dx = 2\operatorname{sh}u\operatorname{ch}u du$.

$J=\int\sqrt{\frac{\operatorname{sh}^2 u+1}{\operatorname{sh}^2 u}}2\operatorname{sh}u\operatorname{ch}u du = \int\frac{\operatorname{ch}u}{\operatorname{sh}u}2\operatorname{sh}u\operatorname{ch}u du = 2\int\operatorname{ch}^2 u du = \frac{1}{2}\int(e^{2u}+2+e^{-2u})du = \frac{1}{4}e^{2u}+u-\frac{1}{4}e^{-2u}+C = \frac{1}{4}e^{2\operatorname{Arsh}\sqrt{x}}+\operatorname{Arsh}\sqrt{x}-\frac{1}{4}e^{-2\operatorname{Arsh}\sqrt{x}}+C = \frac{1}{4}(\sqrt{x+1}+\sqrt{x})^2+\ln(\sqrt{x+1}+\sqrt{x})-\frac{1}{4}(\sqrt{x+1}+\sqrt{x})^{-2}+C == \frac{1}{4}(\sqrt{x+1}+\sqrt{x})^2+\ln(\sqrt{x+1}+\sqrt{x})-\frac{1}{4}(\sqrt{x+1}-\sqrt{x})^2+C = \sqrt{x^2+x}+\ln(\sqrt{x+1}+\sqrt{x})+C.$

Integral J možemo rešiti i smenom $\sqrt{\frac{x+1}{x}}=t$, $x=\frac{1}{t^2-1}$, $dx=-\frac{2t}{(t^2-1)^2}dt$, tj.

$J = -\int\frac{2t^2}{(t^2-1)^2}dt = \frac{1}{2}\int(\frac{1}{t+1}-\frac{1}{(t+1)^2}-\frac{1}{t-1}-\frac{1}{(t-1)^2})dt = \frac{1}{2}(\ln(t+1)+\frac{1}{t+1}-\ln(t-1)+\frac{1}{t-1})+C = \frac{1}{2}\ln\frac{t+1}{t-1}+\frac{t}{t^2-1}+C = \frac{1}{2}\ln\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}+x\sqrt{\frac{x+1}{x}}+C = \ln(\sqrt{x+1}+\sqrt{x})+\sqrt{x^2+x}+C.$

Integral možemo izračunati i na sledeći način:

$t=\sqrt{x+1}-\sqrt{x} \Leftrightarrow \sqrt{x+1}=\sqrt{x}+t \Rightarrow x+1=x+2\sqrt{x}t+t^2 \Rightarrow \sqrt{x}=\frac{1-t^2}{2t}$,

$\frac{1}{2\sqrt{x}}dx = \frac{1}{2}\frac{(1-t^2)'t-(1-t^2)t'}{t^2} = \frac{1}{2}\frac{-2t^2-1+t^2}{t^2}dt = \frac{-t^2-1}{2t^2}dt \Rightarrow dx=\sqrt{x}\cdot\frac{-t^2-1}{t^2}dt = \frac{1-t^2}{2t}\cdot\frac{-t^2-1}{t^2}dt.$

$I = \int\frac{\frac{t^4-1}{2t^3}}{1+\frac{1-t^2}{2t}+\frac{1-t^2}{2t}+t}dt = \int\frac{t^4-1}{2t^2(t+1)}dt = \int\frac{(t-1)(t+1)(t^2+1)}{2t^2(t+1)}dt = \int\frac{t^3-t^2+t-1}{2t^2}dt = \frac{1}{2}\Big[\int t dt-\int dt+\int\frac{dt}{t}-\int t^{-2}dt\Big] =$
$= \frac{1}{2}\Big[\frac{t^2}{2}-t+\ln|t|+\frac{1}{t}\Big] = \frac{t^2}{4}-\frac{t}{2}+\frac{\ln|t|}{2}+\frac{1}{2t}+C = \frac{(\sqrt{x+1}-\sqrt{x})^2}{4}-\frac{\sqrt{x+1}-\sqrt{x}}{2}+\frac{\ln(\sqrt{x+1}-\sqrt{x})}{2}+\frac{1}{2(\sqrt{x+1}-\sqrt{x})}+C.$