MATHEMATICS I -May 2008   Novi Sad

1. Determine the eigenvalues of matrix

$$A=\begin{bmatrix} 0 & 1 & 1 & ... & 1\\ 1 & 0 & a & ... & a\\ 1 & a & 0 & ... & a\\ \vdots & & & ... & \\ 1 & a & a & ... & 0 \end{bmatrix}_{n\times n} \qquad (n\geq 3,\quad a\in\mathbb{R}).$$

2. Calculate

$$L=\lim_{n\to\infty}\sqrt[n]{(1+\frac{1}{2})^1\cdot(1+\frac{1}{4})^2\cdot(1+\frac{1}{6})^3\cdot\ldots\cdot(1+\frac{1}{2n})^n}.$$

3. Find the primitive function of:

$$f(x)=\frac{x+\sin x}{1+\cos x}.$$

MATEMATIKA I -maj 2008   Novi Sad

1. Izračunati karakteristične vrednosti matrice

$$A=\begin{bmatrix} 0 & 1 & 1 & ... & 1\\ 1 & 0 & a & ... & a\\ 1 & a & 0 & ... & a\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & a & a & ... & 0 \end{bmatrix}_{n\times n} \qquad (n\geq 3,\quad a\in\mathbb{R}).$$

2. Izračunati graničnu vrednost

$$L=\lim_{n\to\infty}\sqrt[n]{(1+\frac{1}{2})^1\cdot(1+\frac{1}{4})^2\cdot(1+\frac{1}{6})^3\cdot\ldots\cdot(1+\frac{1}{2n})^n}.$$

3. Naći primitivnu funkciju od:

$$f(x)=\frac{x+\sin x}{1+\cos x}.$$

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Rešenje:

1. $$\det(A-\lambda E)=\begin{vmatrix} -\lambda & 1 & 1 & \ldots & 1\\ 1 & -\lambda & x & \ldots & x\\ 1 & x & -\lambda & \ldots & x\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x & x & \ldots & -\lambda \end{vmatrix}=\begin{vmatrix} -\lambda & 0 & 0 & \ldots & 0\\ 1 & -\lambda & x & \ldots & x\\ 1 & x & -\lambda & \ldots & x\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x & x & \ldots & -\lambda \end{vmatrix}+\begin{vmatrix} 0 & 1 & 1 & \ldots & 1\\ 1 & -\lambda & x & \ldots & x\\ 1 & x & -\lambda & \ldots & x\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x & x & \ldots & -\lambda \end{vmatrix}.$$

Razvijajući prvu determinantu po prvoj vrsti i množeći prvu kolonu druge determinante sa $-x$ i dodajući je ostalim kolonama imamo

$$\det(A-\lambda E)=-\lambda\cdot\begin{vmatrix} -\lambda & x & \ldots & x\\ x & -\lambda & \ldots & x\\ \vdots & \vdots & \vdots & \vdots\\ x & x & \ldots & -\lambda \end{vmatrix}+\begin{vmatrix} 0 & 1 & 1 & \ldots & 1\\ 1 & -\lambda-x & 0 & \ldots & 0\\ 1 & 0 & -\lambda-x & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 0 & 0 & \ldots & -\lambda-x \end{vmatrix}$$

Dodajući ostale kolone prvoj kod prve determinante i dodajući prvoj koloni druge determinante ostale prethodno pomnožene sa $\frac{1}{\lambda+x}$, $\lambda\neq -x$, imamo

$$\det(A-\lambda E) = -\lambda \cdot \begin{vmatrix} (n-1)x-\lambda & x & \dots & x \\ (n-1)x-\lambda & -\lambda & \dots & x \\ \vdots & \vdots & \vdots & \vdots \\ (n-1)x-\lambda & x & \dots & -\lambda \end{vmatrix} + \begin{vmatrix} \frac{n-1}{\lambda+x} & 1 & \dots & 1 \\ 0 & -\lambda-x & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda-x \end{vmatrix}$$

$$= -\lambda[(n-1)x-\lambda] \cdot \begin{vmatrix} 1 & x & \dots & x \\ 1 & -\lambda & \dots & x \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x & \dots & -\lambda \end{vmatrix} + \tfrac{n-1}{\lambda+x}(-\lambda-x)^{n-1}$$

$$= -\lambda[(n-1)x-\lambda] \cdot \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & -\lambda-x & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & -\lambda-x \end{vmatrix} + (-1)^{n-1}(n-1)(\lambda+x)^{n-2}$$

$$= -\lambda[(n-1)x-\lambda](-\lambda-x)^{n-2} + (-1)^{n-1}(n-1)(\lambda+x)^{n-2} = (-1)^{n-1}(\lambda+x)^{n-2}[n-1+(n-1)x\lambda-\lambda^2].$$

2. $\ln L = \lim\limits_{n\to\infty} \frac{\ln(1+\frac{1}{2})+2\ln(1+\frac{1}{4})+3\ln(1+\frac{1}{6})+\ldots+n\ln(1+\frac{1}{2n})}{n}$. Ako uzmemo da je $a_n = \ln(1+\frac{1}{2})+2\ln(1+\frac{1}{4})+3\ln(1+\frac{1}{6})+\ldots+n\ln(1+\frac{1}{2n})$ i $b_n = n$, sobzirom da $b_n \to \infty$, $\{b_n\}$ monotono rastući niz, a postoji $\lim\limits_{n\to\infty} \frac{a_n-a_{n-1}}{b_n-b_{n-1}}$, tada postoji i $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ i važi $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{a_n-a_{n-1}}{b_n-b_{n-1}}$, tj. važi Štolcova teorema. Dakle

$$\ln L = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} k\ln(1+\frac{1}{2k}) - \sum\limits_{k=1}^{n-1} k\ln(1+\frac{1}{2k})}{n-(n-1)} = \lim_{n\to\infty} n\ln(1+\frac{1}{2n})$$

$$= \ln \lim_{n\to\infty} \left((1+\frac{1}{2n})^{2n}\right)^{\frac{1}{2}} = \frac{1}{2},$$

tj. $L = e^{\frac{1}{2}}$.

3. Primitivnu funkciju tražimo na bilo kojem intervalu $I \subset ((2k-1)\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$. Parcijalnom integracijom ($u = x+\sin x$, $dv = \frac{1}{1+\cos x}dx$, $du = (1+\cos x)dx$, $v = \int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1}{\sin^2 x}dx - \int \frac{\cos x}{\sin^2 x}dx = -\operatorname{ctg} x - \int \frac{dt}{t^2} = -\operatorname{ctg} x + \frac{1}{\sin x} = \frac{1-\cos x}{\sin x}$), imamo

$$F(x) = \int \frac{x+\sin x}{1+\cos x}dx = (x+\sin x)\cdot\frac{1-\cos x}{\sin x} - \int \frac{1-\cos x}{\sin x}(1+\cos x)dx$$

$$= \frac{(x+\sin x)(1-\cos x)}{\sin x} - \int \sin x dx = \frac{(x+\sin x)(1-\cos x)}{\sin x} + \cos x + C_0 = \frac{x(1-\cos x)}{\sin x} + C.$$