MATHEMATICS I -May 2009   Budva

1. For the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{n\times n}, \quad n \geq 2,$$

   determine A^{-1}.

2. Find

$$L = \lim_{n\to\infty} n^2\left(\frac{1}{(n+1)(n^2+1)} + \frac{1}{(n+2)(n^2+2^2} + ... + \frac{1}{4n^3}\right).$$

3. Let f be the continuous function on the interval $[\frac{\pi}{4}, \frac{\pi}{3}]$ for which $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\sin x dx = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\cos x dx = 0$, holds.

   Show that for $F(x) = \int\limits_{\frac{\pi}{4}}^{x} f(t)\sin t dt$,

   a) $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{F(x)}{\sin^2 x} dx = 0$, holds, and

   b) the equation $F(x) = 0$ has at least one solution on interval $(\frac{\pi}{4}, \frac{\pi}{3})$, and $f(x) = 0$ has at least two solutions.

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1. Izračunati inverznu matricu matrice

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}_{n\times n}, \quad n \geq 2.$$

2. Izračunati graničnu vrednost

$$L = \lim_{n\to\infty} n^2\left(\frac{1}{(n+1)(n^2+1)} + \frac{1}{(n+2)(n^2+2^2} + ... + \frac{1}{4n^3}\right).$$

3. Neka je f neprekidna funkcija nad $[\frac{\pi}{4}, \frac{\pi}{3}]$ takva $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\sin x dx = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\cos x dx = 0$.

   Ako je $F(x) = \int\limits_{\frac{\pi}{4}}^{x} f(t)\sin t dt$, pokazati:

   a) $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{F(x)}{\sin^2 x} dx = 0$;

   b) na intervalu $(\frac{\pi}{4}, \frac{\pi}{3})$ funkcija F ima bar jednu nulu, a funkcija f bar dve nule.

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Rešenje:

1. $\det(A) = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = \begin{vmatrix} n-1 & n-1 & n-1 & \dots & n-1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = (n-1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} =$
$(n-1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix} = (n-1)(-1)^{n-1}$. $A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = (n-2)(-1)^{n-2}$, i slično $A_{ii} = (n-2)(-1)^{n-2}$.

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix} = -(-1)^{n-2}$. Slično $A_{ij} = (-1)^{n-1},\ i \neq j$.

$$A^{-1} = \frac{1}{\det(A)}[A_{ij}]^T = \frac{1}{(n-1)(-1)^{n-1}} \cdot \begin{bmatrix} (-1)^{n-2}(n-2) & (-1)^{n-1} & \dots & (-1)^{n-1} \\ (-1)^{n-1} & (-1)^{n-2}(n-2) & \dots & (-1)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-1} & (-1)^{n-1} & \dots & (-1)^{n-2}(n-2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2-n}{n-1} & (-1)^{n-1} & \dots & (-1)^{n-1} \\ (-1)^{n-1} & \frac{2-n}{n-1} & \dots & (-1)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-1} & (-1)^{n-1} & \vdots & \frac{2-n}{n-1} \end{bmatrix}.$$

2. $a_n = \sum\limits_{k=1}^{n} \frac{n^2}{(n+k)(n^2+k^2)} = \sum\limits_{k=1}^{n} \frac{1}{n^3} \frac{n^2}{(1+\frac{k}{n})(1+(\frac{k}{n})^2)} = \frac{1}{n} \sum\limits_{k=1}^{n} \frac{1}{(1+\frac{k}{n})(1+(\frac{k}{n})^2)}$,

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \int\limits_0^1 \frac{1}{(1+x)(1+x^2)} dx = \frac{1}{2}\int\limits_0^1 \frac{dx}{1+x} + \frac{1}{2}\int\limits_0^1 \frac{1-x}{1+x^2} dx = \frac{1}{2}\ln|1+x|\Big|_0^1 - \frac{1}{4}\ln(1+x^2)\Big|_0^1 + \frac{1}{2}\operatorname{arctg} x\Big|_0^1 = \frac{1}{4}\ln 2 + \frac{\pi}{8}$.

3. $u = F(x) = \int\limits_{\frac{\pi}{4}}^{x} f(t)\sin t dt$, $du = f(x)\sin x$, $dv = \frac{1}{\sin^2 x}$, $v = -\operatorname{ctg} x$

$\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{F(x)}{\sin^2 x} dx = -\operatorname{ctg} x \int\limits_{\frac{\pi}{4}}^{x} f(x)\sin x dx\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{ctg} x f(x)\sin x dx = 0 + \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} f(x)\cos x dx = 0.$

Zbog neprekidnosti funkcije $F(x)$ (i funkcije $\frac{1}{\sin^2 x}$ na $[\frac{\pi}{4}, \frac{\pi}{3}]$) sledi i neprekidnost funkcije $G(z) = \int\limits_{\frac{\pi}{4}}^{z} \frac{F(x)}{\sin^2 x} dx$ na intervalu $[\frac{\pi}{4}, \frac{\pi}{3}]$ i njena diferencijabilnost na $(\frac{\pi}{4}, \frac{\pi}{3})$, kao i da je $G'(z) = \frac{F(z)}{\sin^2 z}$. Iz a) sledi da je $G(\frac{\pi}{3}) = 0$, pa kako je $G(\frac{\pi}{4}) = 0$, ispunjeni su uslovi Rolove teoreme, tj. postoji $\mu \in (\frac{\pi}{4}, \frac{\pi}{3})$, tako da je $G'(\mu) = \frac{F(\mu)}{\sin^2 \mu} = 0$, tj. $F(\mu) = 0$, pa F ima bar jednu nulu na posmatranom intervalu.

Kako je na osnovu uslova zadatka $F(\frac{\pi}{3}) = 0$, i zbog $F(\mu) = 0 = F(\frac{\pi}{4})$ kao i razmatranja sličnog prethodnom, zadovoljeni su uslovi Rolove teoreme, pa njenom primenom na funkciju G respektivno na intervalima $[\frac{\pi}{4}, \mu]$ i $[\mu, \frac{\pi}{3}]$ imamo da postoje $\mu_1 \in (\frac{\pi}{4}, \mu)$ i $\mu_2 \in (\mu, \frac{\pi}{3})$, takvi da $F'(\mu_i) = f(\mu_i)\sin\mu_i = 0$, tj. $f(\mu_i) = 0$, $i = 1, 2$, odnosno f ima bar dve nule na posmatranom intervalu.